Your Signature ______________________

Instructions:
Please write your name on every page.
Maximum time is 3 hours.
Maximum possible score is 60.

Score

1.	(15)	
2.	(20)	
3.	(15)	
4.	(15)	
Total.	(60)	

Extra sheets attached(if any): ______________________

1. An **Elementary family** on Ω is said to be a collection $\mathcal{E}$ of subsets of Ω such that

- $\emptyset \in \mathcal{E}$
- if $E, F \in \mathcal{E}$ then $E \cap F \in \mathcal{E}$
- if $E \in \mathcal{E}$ then E^c is a finite disjoint union of members of $\mathcal{E}$.

Prove: If $\mathcal{E}$ is an elementary family, the collection $\mathcal{A}$ of finite disjoint unions of members of $\mathcal{E}$ is an algebra.

2. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Suppose $\mathcal{F}$ is an algebra such that $\mathcal{A} = \sigma(\mathcal{F})$.

   (a) Suppose μ is a finite measure and if ν is any other measure on $\mathcal{A}$ with $\mu = \nu$ on $\mathcal{F}$, then $\mu = \nu$.

   (b) Suppose μ is a σ-finite on $\mathcal{F}$ (I.e. there exists $\{\Omega_m\} \in \mathcal{F}$ such that $\mu(\Omega_m) < \infty$ and $\cup_{m=1}^{\infty} \Omega_m = \Omega$)

      (i) For every $m \geq 1$. Define $\mathcal{F} |_{\Omega_m} = \{F \cap \Omega_m : F \in \mathcal{F}\}$ and $\mathcal{A} |_{\Omega_m} = \{A \cap \Omega_m : A \in \mathcal{A}\}$. Show that $\sigma(\mathcal{F} |_{\Omega_m}) = \mathcal{A} |_{\Omega_m}$, where the first term in the equality is a σ-algebra in Ω_m.

      (ii) If ν is any other measure on $\mathcal{A}$ with $\mu = \nu$ on $\mathcal{F}$, then show that $\mu = \nu$ on $\mathcal{A} |_{\Omega_m}$ and consequently $\mu = \nu$ on $\mathcal{A}$.

3. Let Ω be a non-empty set and $\mathcal{A}$ be the σ-field on Ω generated by $\{A\}$ where A is a non-empty proper subset of Ω. Show that a function $f : \Omega \to \mathbb{R}$ is measurable iff $f = \alpha 1_A + \beta 1_{A^c}$ for some $\alpha, \beta \in \mathbb{R}$

4. Let $(\Omega, \mathcal{B}, P)$ be a probability space. Let $X : \Omega \to \mathbb{R}$ be a random variable.

(a) Define what is meant by saying "X is a simple random variable" and in such a case what is $\int X dP$?

(b) Define $\int X dP$ for any random variable X.

(c) Consider $\Omega = (0, 1], \mathcal{B}$ to the Borel σ-algebra and P to be the Lebesgue measure. Let

$$X_n(\omega) = \sum_{k=0}^{2^n - 1} \frac{k}{2^n} 1_{(\frac{k}{2^n}, \frac{k+1}{2^n}]}(\omega)$$

Does X_n converge pointwise and if yes then identify the limit X. Then proceed to evaluate $\int X dP$.